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On the fundamental matrix of the inverse of a polynomial matrix and applications to ARMA representations

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A B S T R A C T

The fundamental matrix of the inverse of a particular Toeplitz structured matrix pencil is used in order to provide an algorithm for the computation of the fundamental matrix of the inverse of a polynomial matrix. The dynamical interpretation of this pencil is explored and its fundamental matrix is used to provide closed formulas for the solutions of AutoRegressive Moving Average (ARMA) representations. Finally, using this pencil the notion of reachability for descriptor state space systems is extended to ARMA representations.

1. Introduction

Transforming polynomial matrices to matrix pencils is a standard way of treating polynomial matrix problems. Matrix pencils are in general easier to manipulate, and provide better insight on the underlying problem. Matrix pencils have been used extensively to compute the eigenvalues of a polynomial matrix [1–3], on the computation of generalized inverse systems [4] and many other control problems. We will use this approach to compute the fundamental matrix sequence of the inverse of a polynomial matrix, study its properties and provide new tools for the analysis of ARMA representations.

Specifically, in the first section, given a regular polynomial matrix $A(z) = A_0 + A_1 z + \cdots + A_q z^q \in \mathbb{R}[z]^{r\times r}$, we introduce a new matrix pencil $P(z) = z\widetilde{E} + \widetilde{A} \in \mathbb{R}[z]^{qr\times qr}$, where the matrices $\widetilde{E}, \widetilde{A}$ are of block Toeplitz structure and determined in terms of the coefficient matrices A_i. We connect the

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fundamental matrix sequences of $A(z)^{-1}$ and $P(z)^{-1}$ and propose a new algorithm for the computation of the fundamental matrix sequence $\{H_i, i = -\hat{q}_r, -\hat{q}_r + 1, \ldots\}$ of the inverse of $A(z)$ i.e. $A(z)^{-1} = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-1}$, by extending the respective results presented in [5,6] to the polynomial matrix case. Thus, the problem of computing the fundamental matrix sequence of $A(z)^{-1}$ is reduced to the equivalent pencil problem and the properties of the fundamental matrix sequence presented for matrix pencils in [5,7], are extended to the general case of polynomial matrices.

In the second section we propose an application of the fundamental matrix sequence of a regular polynomial matrix to the solution of a discrete time AutoRegressive (ARMA) representation. More specifically we reestablish a closed formula that was proved in [8] concerning the forward solution (the solution in terms of the input sequence and the initial conditions) of a discrete time ARMA representation, by extending the results given for discrete time descriptor representations given by [9,10,7]. The proofs presented in this paper provide better insight to the problem instead of the Z-transform method that was used in [8].

Finally, in the last section, we give a new criterion for reachability of discrete time ARMA representations, following the definition of reachability given in [11] and thus extending the criterion of reachability for discrete time descriptor representations given in [7].

2. The Laurent series expansion of a polynomial matrix

Let us consider a regular matrix pencil $zE - A$ i.e. $\det(z_0E - A) \neq 0$ for some $z_0 \in \mathbb{C}$. Then there exists $R_a > 0$ and $|z| > R_a$ for which the Laurent series expansion about infinity for the resolvent matrix is given by

$$(zE - A)^{-1} = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}, \tag{1}$$

where μ is the index of nilpotence. The sequence Φ_i is known as the (forward) fundamental matrix sequence of $(zE - A)^{-1}$. The following properties of the fundamental matrix are well known [5]:

Theorem 1. *With $(zE - A)$ regular and Φ_i defined by (1):*

1. $\Phi_i E - \Phi_{i-1} A = I\delta_i$
2. $E\Phi_i - A\Phi_{i-1} = I\delta_i$
3. $\Phi_i = \begin{Bmatrix} (\Phi_0 A)^i \Phi_0, & i \geqslant 0 \\ (-\Phi_{-1}E)^{-i-1}\Phi_{-1}, & i < 0 \end{Bmatrix}$
4. $\Phi_i E \Phi_j = \Phi_j E \Phi_i$
5. $\Phi_i E \Phi_j = \begin{Bmatrix} -\Phi_{i+j}, & i < 0,\ j < 0 \\ \Phi_{i+j}, & i \geqslant 0,\ j \geqslant 0 \\ 0, & \text{otherwise} \end{Bmatrix}$
6. $\Phi_i A \Phi_j = \begin{Bmatrix} -\Phi_{i+j+1}, & i < 0, j < 0 \\ \Phi_{i+j+1}, & i \geqslant 0, j \geqslant 0 \\ 0, & \text{otherwise} \end{Bmatrix}$

   where δ_i is the Kronecker delta.

Explicit formulas for the coefficients Φ_i has been given in [9,12,10,6,13].

Consider the polynomial matrix

$$A(z) = A_0 + A_1 z + \cdots + A_q z^q \in \mathbb{R}[z]^{r \times r}.$$

By assuming that $A(z)$ is regular i.e. $\det A(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. Then there exists $R_b > 0$ and $|z| > R_b$ for which the Laurent series expansion about infinity for the resolvent matrix $A(z)^{-1}$ is given by:

$$A(z)^{-1} = H_{\hat{q}_r} z^{\hat{q}_r} + H_{\hat{q}_r - 1} z^{\hat{q}_r - 1} + \cdots = \sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i},$$

where $\hat{q}_r$ is the greatest among the orders of the zeros at infinity of $A(z)$. The sequence H_i is known as the (forward) fundamental matrix sequence of $A(z)^{-1}$.

Several authors have tackled the problem of the computation of the Laurent series expansion about infinity for the resolvent matrix of $(zE - A)^{-1}$ [9,12,10,6,13] both theoretically and numerically. In contrast, for the general case of polynomial matrices there exists only one algorithm which has been presented in [14] which is mainly of theoretical importance. In the following we will provide an algorithm that enables us to take advantage of advanced matrix pencils methods for the computation of the Laurent series expansion about infinity for the resolvent matrix $A(z)^{-1}$.

We introduce the following matrix pencil $P(z) = z\widetilde{E} + \widetilde{A} \in \mathbb{R}[z]^{qr\times qr}$ where

$$\widetilde{E} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \in \mathbb{R}^{qr\times qr},$$

$$\widetilde{A} = \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \in \mathbb{R}^{qr\times qr}. \tag{2}$$

Note the Toeplitz structure of $\widetilde{E}$ and $\widetilde{A}$.

Denote also by Φ_i the coefficients of the Laurent series expansion about infinity of the inverse of the resolvent $(z\widetilde{E} + \widetilde{A})^{-1}$ defined by:

$$(z\widetilde{E} + \widetilde{A})^{-1} = \Phi_{-\mu} z^{\mu-1} + \Phi_{-\mu+1} z^{\mu-2} + \cdots = z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}. \tag{3}$$

The next theorem connects the fundamental matrix sequence of $A(z)^{-1}$ and $(z\widetilde{E} + \widetilde{A})^{-1}$.

Theorem 2. *The fundamental matrix sequences H_i of $A(z)^{-1}$ and Φ_i of $(z\widetilde{E} + \widetilde{A})^{-1}$ are connected by*:

$$\Phi_i = \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}. \tag{4}$$

Proof. Since H_i are coefficients of the Laurent series expansion of $A(z)^{-1}$ we have that

$$A(z)\,A(z)^{-1} = I_r \Leftrightarrow \left(\sum_{i=0}^{q} A_i z^i\right)\left(\sum_{i=-\hat{q}_r}^{\infty} H_i z^{-i}\right) = I_r \Leftrightarrow$$
$$\sum_{i=0}^{q} A_i H_{i-k} = \delta_k I_r \left(\text{or} \sum_{i=0}^{q} H_{i-k} A_i = \delta_k I_r\right) \tag{5}$$

Using the above relation, we can establish the following relation

$$(z\widetilde{E} + \widetilde{A})\left(z^{-1} \sum_{i=-\mu}^{\infty} \Phi_i z^{-i}\right) = I_{qr}$$

or equivalently

$$\widetilde{E}\Phi_i + \widetilde{A}\Phi_{i-1} = \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix} + \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{q-1} & A_{q-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} H_{-qi} & H_{-qi-1} & \cdots & H_{-q-qi+1} \\ H_{-qi+1} & H_{-qi} & \cdots & H_{-q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi+q-1} & H_{-qi+q-2} & \cdots & H_{-qi} \end{bmatrix} \overset{(5)}{=} I_{qr}\delta_i$$

Note that we have replaced the matrices E, A in (1) with $\widetilde{E}, -\widetilde{A}$ in (3). $\square$

Theorem 2 leads to the following algorithm for the computation of the coefficients of the Laurent series expansion at infinity of $A(z)^{-1}$.

Step 1. Construct the matrices $\widetilde{E}, \widetilde{A}$ defined in (2).

Step 2. Determine the matrices Φ_0, Φ_{-1} of the resolvent $(z\widetilde{E} + \widetilde{A})^{-1}$ by using one of the known computing techniques described in [9,12,10,6,13]. Compute the coefficients $H_{-2q+1}, \ldots, H_{q-1}$ from the following relations

$$\Phi_0 = \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix},$$

$$\Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} & \cdots & H_{-q+1} \\ H_1 & H_0 & \cdots & H_{-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{q-1} & H_{q-2} & \cdots & H_0 \end{bmatrix}.$$

Step 3. The rest of the terms can be determined by using the property (3) of Theorem 1

$$\Phi_i = \begin{Bmatrix} \left(-\Phi_0\widetilde{A}\right)^i \Phi_0, i \geqslant 0 \\ \left(-\Phi_{-1}\widetilde{E}\right)^{-i-1} \Phi_{-1}, i < 0 \end{Bmatrix}$$

$$= \begin{bmatrix} H_{-q-qi} & H_{-q-qi-1} & \cdots & H_{-2q-qi+1} \\ H_{-q-qi+1} & H_{-q-qi} & \cdots & H_{-2q-qi+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qi-1} & H_{-qi-2} & \cdots & H_{-qi-q} \end{bmatrix}.$$

Since Φ_i are the coefficients of the Laurent expansion at infinity of $(z\widetilde{E} + \widetilde{A})^{-1}$, corresponding properties to those defined in Theorem 1 can now be established for polynomial matrices by extending the results in [7].

Theorem 3. *With $A(z)$ regular and Φ_i defined by (3) and (4):*

1. $\Phi_i\widetilde{E} + \Phi_{i-1}\widetilde{A} = I\delta_i$

2. $\widetilde{E}\Phi_i + \widetilde{A}\Phi_{i-1} = I\delta_i$
3. $\Phi_i = \begin{Bmatrix} (-\Phi_0\widetilde{A})^i\Phi_0,\ i \geqslant 0 \\ (-\Phi_{-1}\widetilde{E})^{-i-1}\Phi_{-1},\ i < 0 \end{Bmatrix}$
4. $\Phi_i\widetilde{E}\Phi_j = \Phi_j\widetilde{E}\Phi_i$
5. $\Phi_i\widetilde{E}\Phi_j = \begin{Bmatrix} -\Phi_{i+j},\ i < 0, j < 0 \\ \Phi_{i+j},\ i \geqslant 0, j \geqslant 0 \\ 0,\ \text{otherwise} \end{Bmatrix}$
6. $\Phi_i\widetilde{A}\Phi_j = \begin{Bmatrix} -\Phi_{i+j+1},\ i < 0,\ j < 0 \\ \Phi_{i+j+1},\ i \geqslant 0,\ j \geqslant 0 \\ 0,\ \text{otherwise} \end{Bmatrix}$.

Example 1. Consider the polynomial matrix

$$A(z) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} z^2$$

We will construct the fundamental matrix sequence of $A(z)^{-1}$.

Step 1. Construct the matrices $\widetilde{E}, \widetilde{A}$ defined in (2):

$$\widetilde{E} = \begin{bmatrix} A_2 & A_1 \\ 0 & A_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \widetilde{A} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Step 2. Determine the matrices Φ_0, Φ_{-1} of the resolvent $(z\widetilde{E} + \widetilde{A})^{-1}$ using the algorithm presented in [13]:

$$\Phi_0 = \begin{bmatrix} H_{-2} & H_{-3} \\ H_{-1} & H_{-2} \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

$$\Phi_{-1} = \begin{bmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore

$$H_{-3} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}, \quad H_{-2} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$H_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_0 = H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 3. The rest terms can be determined by using the property (3) of Theorem 1

$$\Phi_i = \begin{Bmatrix} (-\Phi_0\widetilde{A})^i\,\Phi_0,\ i \geqslant 0 \\ (-\Phi_{-1}\widetilde{E})^{-i-1}\,\Phi_{-1},\ i < 0 \end{Bmatrix} = \begin{bmatrix} H_{-2i-2} & H_{-2i-3} \\ H_{-2i-1} & H_{-2i-2} \end{bmatrix}.$$

For example

$$\Phi_1 = (-\Phi_0\widetilde{A})^1\,\Phi_0 = \begin{bmatrix} H_{-4} & H_{-5} \\ H_{-3} & H_{-4} \end{bmatrix} = \left[\begin{array}{cc|cc} -4 & -3 & 7 & 6 \\ 1 & 1 & -2 & -2 \\ \hline 2 & 2 & -4 & -3 \\ -1 & 0 & 1 & 1 \end{array}\right].$$

3. On the solution of a discrete time ARMA representation

AutoRegressive Moving Average (ARMA) representations are a flexible and powerful modeling tool applicable in a variety of scientific areas such as the analysis of circuits [15], economics [16], power systems [17], etc. In this section we will examine the connection of the ARMA representation described by a polynomial matrix $A(z)$ and the corresponding descriptor system described by the matrix pencil $P(z)$. We will provide a closed formula for the computation of the forward solution of the ARMA system and necessary and sufficient conditions for its existence. It is necessary to note here that such forward solutions have already been established in [8,18] after many pages of matrix manipulations by using Z-transforms. In the paragraphs below, we will provide a more illuminating proof by using the corresponding descriptor system described by the matrix pencil $P(z)$.

Consider the following descriptor system

$$Ex_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \ldots, N-1 \tag{6}$$

with $x_k : [0, N] \to \mathbb{R}^n, u_k : [0, N] \to \mathbb{R}^m$ and the pencil $zE - A$ assumed to be regular.

There have been several interpretations of Eq. (6). From a dynamical standpoint we may consider that the initial condition x_0 is given and that is desired to determine the state x_k in a forward fashion from the input sequence and the previous values of the semistate. We call this the *forward solution*[1] of (6) and is given in [7] by:

$$x_k = \Phi_k E x_0 + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} B u_i, \tag{7}$$

whereas a necessary and sufficient condition for the descriptor system (6) to have a forward solution, is

$$x_0 = \Phi_0 E x_0 + \sum_{i=0}^{\mu-1} \Phi_{-i-1} B u_i. \tag{8}$$

Consider a linear, time invariant discrete time system, described by the difference equation:

$$A_0 y_k + A_1 y_{k+1} + \cdots + A_q y_{k+q} = B_0 u_k + \cdots + B_q u_{k+q},$$

with $k = 0, 1, \ldots, N-q$ and $|A(\sigma)| \neq 0$, or otherwise

$$A(\sigma) y_k = B(\sigma) u_k, \tag{9}$$

where σ denotes the shift-forward operator, $y_k : [0, N] \to \mathbb{R}^r$ is the output of the system, $u_k : [0, N] \to \mathbb{R}^m$ is a known input of the system, and

$$A(\sigma) = A_0 + A_1\sigma + \cdots + A_q\sigma^q \in \mathbb{R}[\sigma]^{r\times r},$$
$$B(\sigma) = B_0 + B_1\sigma + \cdots + B_q\sigma^q \in \mathbb{R}[\sigma]^{r\times m}.$$

The above description is also known as the AutoRegressive Moving Average (ARMA) representation of a system. We may rewrite the above equations for $k = 0, 1, \ldots, \left[\frac{N}{q}\right] - 1$, where $[N/q]$ denotes the integer part of the rational number N/q as follows:

$$\widetilde{E}x_{k+1} + \widetilde{A}x_k = \widetilde{B}v_k, \quad k = 0, 1, \ldots, \left[\frac{N}{q}\right] - 1, \tag{10}$$

where $\widetilde{E}, \widetilde{A} \in \mathbb{R}^{qr\times qr}$ defined in (2),

[1] Other different interpretations of the descriptor system give rise to closed formulas [7] for *backward* and *symmetric solution* of (6).

$$\widetilde{B} = \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \in \mathbb{R}^{qr \times 2qr}$$

and

$$x_k = \begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix}, \quad v_k = \begin{bmatrix} u_{kq+2q-1} \\ u_{kq+2q-2} \\ \vdots \\ u_{kq+0} \end{bmatrix}. \tag{11}$$

Eq. (11) provides a mapping between the solutions of (9) and those of (10). Since N is usually not a multiple of the number q, we can always extend our interval $[0, N]$ to $[0, \widehat{N}]$ where $\widehat{N} = nq$, by including new states or omitting some of the last states of the system, always based on the initial conditions of the system and the solution of the system that will describe in the sequel. Therefore we assume in what follows, that $N = \widehat{N} = nq$.

Applying the forward solution formula for singular systems described in (7) to the system (10) we have that

$$\begin{bmatrix} y_{kq+q-1}^T & y_{kq+q-2}^T & \cdots & y_{kq+0}^T \end{bmatrix}^T = x_k = \Phi_k \widetilde{E} x_0 + \sum_{i=0}^{k+\mu-1} \Phi_{k-i-1} \widetilde{B} v_i$$

or equivalently by using the fundamental matrix sequence of $A(z)^{-1}$, and Theorem 2

$$\begin{bmatrix} H_{-q-qk} & H_{-q-qk-1} & \cdots & H_{-2q-qk+1} \\ H_{-q-qk+1} & H_{-q-qk} & \cdots & H_{-2q-qk+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-qk-1} & H_{-qk-2} & \cdots & H_{-qk-q} \end{bmatrix} \begin{bmatrix} A_q & A_{q-1} & \cdots & A_1 \\ 0 & A_q & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_q \end{bmatrix} \begin{bmatrix} y_{q-1} \\ y_{q-2} \\ \vdots \\ y_0 \end{bmatrix}$$
$$+ \sum_{i=0}^{k+\mu-1} \begin{bmatrix} H_{-qk+qi} & \cdots & H_{-qk+qi-q+1} \\ H_{-qk+qi+1} & \cdots & H_{-qk+qi-q+2} \\ \vdots & \ddots & \vdots \\ H_{-qk+qi+q-1} & \cdots & H_{-qk+qi} \end{bmatrix}$$
$$\times \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{iq+2q-1} \\ u_{iq+2q-2} \\ \vdots \\ u_{iq+0} \end{bmatrix}.$$

So the forward solution of (10) for $k = 0, 1, \ldots, \left[\frac{N}{q}\right] - 1$ is

$$x_k = \begin{bmatrix} y_{kq+q-1} \\ y_{kq+q-2} \\ \vdots \\ y_{kq+0} \end{bmatrix}$$
$$= \begin{bmatrix} H_{-2q-qk+1} & \cdots & H_{-q-qk-1} & H_{-q-qk} \\ H_{-2q-qk+2} & \cdots & H_{-q-qk} & H_{-q-qk+1} \\ \vdots & \ddots & \vdots & \vdots \\ H_{-qk-q} & \cdots & H_{-qk-2} & H_{-qk-1} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}$$

$$
+\begin{bmatrix} H_{-qk-q+1} & \cdots & H_{q\hat{q}_r-q} & H_{q\hat{q}_r-q+1} \\ H_{-qk-q+2} & \cdots & H_{q\hat{q}_r-q+1} & H_{q\hat{q}_r-q+2} \\ \vdots & \ddots & \vdots & \vdots \\ H_{-qk} & \cdots & H_{q\hat{q}_r-1} & H_{q\hat{q}_r} \end{bmatrix}
$$

$$
\times \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{(k+\hat{q}_r+q)q-1} \\ u_{(k+\hat{q}_r+q)q} \end{bmatrix}.
$$

Taking into account that y_k corresponds to the $(k \bmod q+1)$ line of $x_{\lceil \frac{k}{q} \rceil}$ as computed above or otherwise replacing kq with k in the last of the above q equations, we get a formula for the computation of the forward solution of the ARMA system (9).

$$
y_k = \begin{bmatrix} H_{-k-q} & H_{-k-q+1} & \cdots & H_{-k-1} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}
$$

$$
+\begin{bmatrix} H_{-k} & H_{-k+1} & \cdots & H_{\hat{q}_r} \end{bmatrix} \begin{bmatrix} B_0 & B_1 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & B_1 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k+\hat{q}_r+q} \end{bmatrix}, \tag{12}
$$

where $k = 0, 1, \ldots, \widehat{N}$.

The necessary and sufficient conditions for the existence of solution of (10) become using Theorem 2

$$
x_0 = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} H_{-q} & H_{-q-1} & \cdots & H_{-2q+1} \\ H_{-q+1} & H_{-q} & \cdots & H_{-2q+2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-1} & H_{-2} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}
$$

$$
+\sum_{i=0}^{\mu-1} \begin{bmatrix} H_{qi} & \cdots & H_{qi-q+1} \\ H_{qi+1} & \cdots & H_{qi-q+2} \\ \vdots & \ddots & \vdots \\ H_{qi+q-1} & \cdots & H_{qi} \end{bmatrix} \begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} \begin{bmatrix} u_{iq+2q-1} \\ u_{iq+2q-2} \\ \vdots \\ u_{iq+0} \end{bmatrix}
$$

and by rewriting the sum

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} H_{-q} & H_{-q+1} & \cdots & H_{-1} \\ H_{-q-1} & H_{-q} & \cdots & H_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{-2q+1} & H_{-2q+2} & \cdots & H_{-q} \end{bmatrix} \begin{bmatrix} A_q & 0 & \cdots & 0 \\ A_{q-1} & A_q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_q \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix}$$
$$+ \begin{bmatrix} H_0 & \cdots & H_{\hat{q}_r} & 0 & \cdots & 0 \\ H_{-1} & \cdots & H_{\hat{q}_r-1} & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & H_{\hat{q}_r-q+2} & \cdots & H_{\hat{q}_r} \end{bmatrix}$$
$$\times \begin{bmatrix} B_0 & \cdots & B_q & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & B_q & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_0 & B_1 & \cdots & B_q \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{2q-1+\hat{q}_r} \end{bmatrix}. \tag{13}$$

Eq. (13) are considered to be necessary and sufficient conditions for the ARMA system (9) to have a solution.

Therefore, we have established a connection between the forward dynamical behavior of two systems (10) and (9). Using similar techniques, the backward and symmetric solution for the ARMA-representation (9) can be deduced using the closed formulas found for descriptor systems in [7].

Example 2. Consider the following discrete time ARMA representation:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{A_0} y_k + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} y_{k+1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_2} y_{k+2} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_0} u_k \tag{14}$$

or equivalently

$$\underbrace{\begin{bmatrix} \sigma+1 & \sigma-1 \\ 1 & \sigma^2 \end{bmatrix}}_{A(\sigma)} \underbrace{\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix}}_{y_k} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B(\sigma)} u_k$$

Then from the previous example we have that

$$\Phi_0 = \widehat{E}^D \left(c\widetilde{E} + \widetilde{A}\right)^{-1} = \left[\begin{array}{cc|cc} -1 & -1 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} H_{-2} & H_{-3} \\ H_{-1} & H_{-2} \end{bmatrix} \tag{15}$$

and

$$\Phi_{-1} = -\left(I_4 - \widehat{E}\widehat{E}^D\right)\widehat{A}^D \left(c\widetilde{E} + \widetilde{A}\right)^{-1} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} H_0 & H_{-1} \\ H_1 & H_0 \end{bmatrix}. \tag{16}$$

Using property (3) of Theorem 3 we get that

$$\Phi_1 = \begin{bmatrix} H_{-4} & H_{-5} \\ H_{-3} & H_{-4} \end{bmatrix} = -\Phi_0 \widetilde{A} \Phi_0 = \left[\begin{array}{cc|cc} -4 & -3 & 7 & 6 \\ 1 & 1 & -2 & -2 \\ \hline 2 & 2 & -4 & -3 \\ -1 & 0 & 1 & 1 \end{array}\right]. \tag{17}$$

Using (15), (16), (17) we conclude that

$$H_{-5}=\begin{bmatrix} 7 & 6 \\ -2 & -2 \end{bmatrix}; \quad H_{-4}=\begin{bmatrix} -4 & -3 \\ 1 & 1 \end{bmatrix}; \quad H_{-3}=\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix};$$
$$H_{-2}=\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}; \quad H_{-1}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad H_0=H_1=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The forward solution of the ARMA system is given by (12) and so

$$y_2=\begin{bmatrix} H_{-4} & H_{-3} \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}+\begin{bmatrix} H_{-2} & H_{-1} \end{bmatrix}\begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ 0 & B_0 & B_1 & B_2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

which after some operations becomes

$$y_2=\begin{bmatrix} -u_0+2y_{1,0}-y_{2,0}+2y_{2,1} \\ -y_{1,0}+u_0 \end{bmatrix}$$

Similarly, we get

$$y_3=\begin{bmatrix} H_{-5} & H_{-4} \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$
$$+\begin{bmatrix} H_{-3} & H_{-2} & H_{-1} \end{bmatrix}\begin{bmatrix} B_0 & B_1 & B_2 & 0 & 0 \\ 0 & B_0 & B_1 & B_2 & 0 \\ 0 & 0 & B_0 & B_1 & B_2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

which becomes

$$y_3=\begin{bmatrix} 2u_0-u_1-4y_{1,0}+2y_{2,0}-3y_{2,1} \\ u_1+y_0-y_{2,0}+y_{2,1} \end{bmatrix}$$

According to (13), a necessary and sufficient condition for the ARMA-representation (14) to have a forward solution is

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}:=\begin{bmatrix} H_{-2} & H_{-1} \\ H_{-3} & H_{-2} \end{bmatrix}\begin{bmatrix} A_2 & 0 \\ A_1 & A_2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$
$$+\begin{bmatrix} H_0 & H_1 \\ H_{-1} & H_0 \end{bmatrix}\begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ 0 & B_0 & B_1 & B_2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} y_{1,0} \\ y_{2,0} \\ -y_{1,0}+y_{2,0}-y_{2,1} \\ y_{2,1} \end{bmatrix},$$

i.e.

$$y_{1,1}=-y_{1,0}+y_{2,0}-y_{2,1}.$$

4. Reachability of discrete-time ARMA representations

In [7], reachability was defined for descriptor systems, whereas in [11] an extension of this definition to monic ARMA representations was given. Following the aforementioned definitions and based on the observation that reachability of (9) is equivalent to the output reachability of (10), we propose a new criterion for reachability of non monic ARMA representations. The definition of system properties depends on whether we are interested on the forward, backward or symmetric solution. We consider here only the forward case, for descriptor systems and ARMA representations.

Before proceeding to the main results, we will need the following lemma.

Lemma 4. *The degrees of the determinants of $z\widetilde{E}+\widetilde{A}$ and $A(z)$ are equal i.e.*

$$\deg\left[\det\left(z\widetilde{E}+\widetilde{A}\right)\right]=\deg\left[\det A\left(z\right)\right]. \tag{18}$$

Proof. See the discussion in the Appendix. □

We continue with the definition of reachability for descriptor systems.

Definition 1 [7]. The descriptor system

$$\begin{aligned}&\widetilde{E}x_{k+1}+\widetilde{A}x_k=\widetilde{B}v_k\\&z_k=\widetilde{C}x_k\\&\widetilde{E},\widetilde{A}\in\mathbb{R}^{r\times r},\quad \widetilde{B}\in\mathbb{R}^{r\times m},\quad \widetilde{C}\in\mathbb{R}^{p\times r}\end{aligned} \tag{19}$$

is reachable in the forward sense if for $x_0=0\in\mathbb{R}^r$ and each $\xi_1\in\mathbb{R}^r$, there exists a control function $u_{0,k+\mu}$ for some $k>0$ such that $x_k=\xi_1$.

Let $\{\Phi_i, i=-\mu,-\mu+1,\ldots\}$ be the forward fundamental matrix sequence, of $z\widetilde{E}+\widetilde{A}$, then [7] has presented the following criterion for forward reachability.

Theorem 5. *The descriptor system* (19) *is reachable in the forward sense if and only if*

$$rank_{\mathbb{R}}\left(U_{\tilde{r}}\right)=r$$

where the forward reachability matrix is defined as

$$U_k\equiv\left(\Phi_{-\mu}\widetilde{B}\quad\cdots\quad\Phi_{-1}\widetilde{B}\quad\Phi_0\widetilde{B}\quad\cdots\quad\Phi_{k-1}\widetilde{B}\right),$$

with

$$\tilde{r}\equiv\deg\left(\det\left(z\widetilde{E}+\widetilde{A}\right)\right).$$

Since, we are not interested to control every sequence of q states of (10), but only the last element of every q-tuple i.e. $\left(0\quad 0\quad\cdots\quad I\right)x_k$, we need to define the output reachability in a similar manner using the definition of reachability given above.

Definition 2. The descriptor system (19) is output reachable in the forward sense if for $x_0=0\in\mathbb{R}^r$ (and therefore $z_0=0\in\mathbb{R}^p$) and each $\xi_1\in\mathbb{R}^p$, there exists a control function $u_{0,k+\mu}$ for some $k>0$ such that $z_k=\xi_1$.

Following similar lines with the work of [7] we can give a criterion for output reachability in the forward sense.

Theorem 6. *The descriptor system* (19) *is output reachable in the forward sense if and only if*

$$rank_{\mathbb{R}}\left(\widetilde{C}U_{\tilde{r}}\right)=p.$$

Proof. According to the forward solution x_k of (19), the output z_k is

$$\begin{aligned}z_k&=\widetilde{C}\Phi_k\widetilde{E}x_0+\sum_{i=0}^{k+\mu-1}\widetilde{C}\Phi_{k-i-1}\widetilde{B}u_i\\&\overset{x_0=0}{=}\sum_{i=0}^{k+\mu-1}\widetilde{C}\Phi_{k-i-1}\widetilde{B}u_i=\widetilde{C}U_k u_{0,k+\mu},\end{aligned}$$

where $u_{0,k+\mu} = (u_{k+\mu-1} \quad u_{k+\mu-2} \quad \cdots \quad u_0)$. System (19) is output reachable in the forward sense, if and only if, for some $k > 0$

$$rank_{\mathbb{R}} \left(\widetilde{C}\Phi_{-\mu}\widetilde{B} \quad \cdots \quad \widetilde{C}\Phi_{-1}\widetilde{B} \quad \widetilde{C}\Phi_0\widetilde{B} \quad \cdots \quad \widetilde{C}\Phi_{k-1}\widetilde{B}\right) = p. \tag{20}$$

Using (18), the determinant of $z\widetilde{E} + \widetilde{A}$ can be written as

$$q(z) = \det\left(z\widetilde{E} + \widetilde{A}\right) = \sum_{i=0}^{\tilde{r}} q_i z^i.$$

Also, the fundamental matrix satisfies the generalized Cayley-Hamilton theorem

$$\sum_{i=0}^{\tilde{r}} q_i \Phi_{k-r+i} = 0 \text{ for} k \geqslant \tilde{r}$$

Therefore (20) may be terminated at $k = \tilde{r}$. $\square$

In case where the system is output reachable in the forward sense, then we can reach any output $\xi_1 \in \mathbb{R}^p$, from an initial condition $x_0 \neq 0$, by using the control input

$$u_{0,\tilde{r}+\mu} = \left(\widetilde{C}U_{\tilde{r}}\right)^T \left[\left(\widetilde{C}U_{\tilde{r}}\right)\left(\widetilde{C}U_{\tilde{r}}\right)^T\right]^{-1} \left\{\xi_1 - \widetilde{C}\Phi_{\tilde{r}}\widetilde{E}x_0\right\}.$$

Carriegos et al. [11] provides a notion of reachability for a special class of ARMA-representations with unit leading coefficient matrix i.e. $A_q = I_r$. Note, that in this last case $(A_q = I_r)$ the polynomial matrix does not have an infinite zero structure, and therefore may possess different properties with the ones which we study here (like descriptor systems have different structural properties with state space systems). In the following we generalize [11] and extend reachability notions and criteria of descriptor systems [7] to ARMA-representations.

Definition 3. The ARMA representation (9) is reachable in the forward sense if for each $y_{init} = (y_0, y_1, .., y_{q-1}) \equiv 0_{rq}$ and $z_1 \in \mathbb{R}^r$, there exists a control function $u_{0,k+\mu}$ for some $k > 0$ such that $y_k = z_1$.

A citerion for reachability in the forward sense, based on the solution (12), is given in the next theorem.

Theorem 7. *Eq.* (9) *is reachable in the forward sense if and only if*

$$rank\left[H_{\mu-1} \quad H_{\mu-2} \quad \cdots \quad H_{-r_1}\right]\begin{bmatrix} B_q & \cdots & B_0 & 0 & \cdots & 0 \\ 0 & B_q & \cdots & B_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0 \end{bmatrix} = r,$$

where $r_1 \equiv \deg(\det A(z))$.

Proof. Consider the system (10)

$$\widetilde{E}x_{k+1} + \widetilde{A}x_k = \widetilde{B}v_k$$

and define

$$z_k \equiv y_{kq} = \underbrace{\begin{bmatrix}0 & 0 & \cdots & I\end{bmatrix}}_{\widetilde{C}} x_k$$

Then the above descriptor system is output reachable in the forward sense according to Theorem 6, if and only if

$$rank\left[\widetilde{C}\Phi_{-\mu}\widetilde{B} \quad \cdots \quad \widetilde{C}\Phi_{-1}\widetilde{B} \quad \widetilde{C}\Phi_0\widetilde{B} \quad \cdots \quad \widetilde{C}\Phi_{\tilde{r}-1}\widetilde{B}\right] = r$$

or equivalently after some operations and by using the generalized Cayley–Hamilton

$$rank\begin{bmatrix}H_{q\mu-1} & H_{q\mu-2} & \cdots & H_{-qr_1}\end{bmatrix}\begin{bmatrix}B_q & \cdots & B_0 & 0 & \cdots & 0\\ 0 & B_q & \cdots & B_0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0\end{bmatrix}=r,$$

where

$$r_1=\deg\left(\det\left(z\widetilde{E}+\widetilde{A}\right)\right)=\deg\left(\det A\left(z\right)\right).$$

However, output reachability in the forward sense of (10) implies the ability to control the state z_{kq}, under zero initial conditions, with the help of the control input $u_{0,k+\mu}$ for some $k>0$. By replacing kq with k we get that (9) is reachable in the forward sense if and only if

$$rank\begin{bmatrix}H_{\mu-1} & H_{\mu-2} & \cdots & H_{-r_1}\end{bmatrix}\begin{bmatrix}B_q & \cdots & B_0 & 0 & \cdots & 0\\ 0 & B_q & \cdots & B_0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & B_q & B_{q-1} & \cdots & B_0\end{bmatrix}=r. \quad \square$$

It is easily seen that the reachability criterion of Theorem 7 reduces to the known criterion of state-space systems if $A(z)=zI-A$ and $B(z)=B_0$ (since $H_{-i}=A^i$), for descriptor systems [7] if $A(z)=zE+A$ and $B(z)=B_0$ (see Theorem 5) or finally for ARMA-representations [11] if $A_q=I_r$.

Example 3. Consider the discrete time ARMA representation of Example 2

$$\underbrace{\begin{bmatrix}z+1 & z-1\\ 1 & z^2\end{bmatrix}}_{A(z)}\underbrace{\begin{bmatrix}y_{1,k}\\ y_{2,k}\end{bmatrix}}_{y_k}=\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B(z)}u_k$$

Since $r_1=\deg(\det A(z))=3$, we have that

$$rank\begin{bmatrix}H_{-1} & H_{-2} & H_{-3}\end{bmatrix}\begin{bmatrix}B_0 & 0 & 0\\ 0 & B_0 & 0\\ 0 & 0 & B_0\end{bmatrix}=rank\begin{bmatrix}0 & -1 & 2\\ 0 & 1 & 0\end{bmatrix}=2$$

and therefore the system is reachable in the forward sense. However, the descriptor system given by (10)

$$\underbrace{\begin{bmatrix}A_2 & A_1\\ 0 & A_2\end{bmatrix}}_{\widetilde{E}}x_{k+1}+\underbrace{\begin{bmatrix}A_0 & 0\\ A_1 & A_0\end{bmatrix}}_{\widetilde{A}}x_k=\underbrace{\begin{bmatrix}B_0 & 0\\ 0 & B_0\end{bmatrix}}_{\widetilde{B}}v_k$$

or equivalently

$$\left[\begin{array}{cc|cc}0 & 0 & 1 & 1\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right]x_{k+1}+\left[\begin{array}{cc|cc}1 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 1 & 1 & 1 & -1\\ 0 & 0 & 1 & 0\end{array}\right]x_k=\left[\begin{array}{c|c}0 & 0\\ 1 & 0\\ \hline 0 & 0\\ 0 & 1\end{array}\right]v_k$$

is not reachable in the forward sense since the reachability matrix has not full row rank i.e.

$$\begin{aligned}&rank\begin{bmatrix}\Phi_{-1}\widetilde{B} & \Phi_0\widetilde{B} & \Phi_1\widetilde{B} & \Phi_2\widetilde{B}\end{bmatrix}\\ &=rank\left[\begin{array}{cc|cc|cc|cc}0 & 0 & -1 & 2 & -3 & 6 & -11 & 20\\ 0 & 0 & 1 & 0 & 1 & -2 & 3 & -6\\ 0 & 0 & 0 & -1 & 2 & -3 & 6 & -11\\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 3\end{array}\right]=3<4,\end{aligned}$$

where $\Phi_0, \Phi_{-1}, \Phi_1$ have been determined in Example 2 and

$$\Phi_2 = \begin{bmatrix} H_{-6} & H_{-7} \\ H_{-5} & H_{-6} \end{bmatrix} = (\Phi_0 A)^2 \Phi_0 = \begin{bmatrix} -13 & -11 & 24 & 20 \\ 4 & 3 & -7 & -6 \\ 7 & 6 & -13 & -11 \\ -2 & -2 & 4 & 3 \end{bmatrix}.$$

It is easily seen from the above example that although the descriptor system (10) is not reachable in the forward sense, the ARMA-representation (9) is reachable. This is because, the reachability of (10) requires that every *sequence* of q states can be reached, but for the reachability of (9) it is sufficient to reach every single state.

5. Conclusions

A matrix pencil $P(z) = z\widetilde{E} + \widetilde{A}$ of particular Toeplitz structure corresponding to a polynomial matrix $A(z) = A_0 + A_1 z + \cdots + A_q z^q \in \mathbb{R}[z]^{r\times r}$ has been presented. We have shown that in order to compute the fundamental matrix sequence of the inverse of a regular polynomial matrix $A(z)$, it is enough to compute the fundamental matrix of the inverse of this augmented matrix pencil $P(z)$. In that way one of the many algorithms applicable only to the matrix pencil case can be used. Also the properties of the fundamental matrix sequence presented for matrix pencils have been extended to the general case of polynomial matrices.

We examined the connection of the ARMA representation described by $A(z)$ and the corresponding descriptor system described by the matrix pencil $P(z)$ and provided a closed loop formula for the computation of the forward solution of the ARMA representation. Also, through the use of $P(z)$, we provide a solid definition of reachability notions and criteria for ARMA representations, by generalizing descriptor system results.

Further research on the subject could address more specific numerical problems, such as the development of a numerical method for the computation of the fundamental matrix sequence of the inverse of a matrix pencil whose coefficients are of Toeplitz structure. Also, using similar techniques as in the forward case presented in this paper, both backward and symmetric solutions and reachability criteria for ARMA representations can be deduced.

Appendix. Proof of Lemma 4

The main purpose of this appendix is to prove that the degrees of the determinants of $z\widetilde{E} + \widetilde{A}$ where $\widetilde{E}$ and $\widetilde{A}$ are defined in (2) and $A(z)$ are equal. We will need the following theorem.

Theorem 8 [19]. *Let*

$$T_i = \begin{bmatrix} A_q & & A_{-i+1} & A_{-i} \\ & \ddots & & \vdots \\ & & A_q & A_{q-1} \\ & & & A_q \end{bmatrix} \in \mathbb{R}^{n(q+i+1)\times n(q+i+1)} \tag{21}$$

denote a Toeplitz matrix built from coefficients of a non-singular $n \times n$ polynomial matrix $A(z)$, where it is assumed that $T_{i-a} = 0$ and $A_i = 0$ when $i < 0$. Let

$$r_i = rankT_i - rankT_{i-1}$$

and

$$k_0 = \min\{i : r_i = n, i \geqslant 0\}.$$

Then the degree δ of the determinant of $A(z)$ is given by

$$\delta = rankT_{k_0} - n(k_0 + 1). \tag{22}$$

In the proof of Theorem 8 in [19], one can notice that Eq. (22) holds for every $k \geqslant k_0$ i.e.

$$\delta = rankT_k - n(k+1), \forall k \geqslant k_0. \tag{23}$$

We will apply Theorem 8 and equation (23) to the matrix pencil $z\widetilde{E} + \widetilde{A}$ of dimension nq.

Theorem 9. *Let*

$$\widetilde{T}_j = \begin{bmatrix} \tilde{E} & \tilde{A} & & \\ & \ddots & \ddots & \vdots \\ & & \tilde{E} & \tilde{A} \\ & & & \tilde{E} \end{bmatrix} \in \mathbb{R}^{nq(j+2)\times nq(j+2)}$$

denote the Toeplitz matrix corresponding to (21) built from coefficients of the $nq \times nq$ pencil $z\widetilde{E} + \widetilde{A}$. Let

$$\tilde{r}_j = rank\widetilde{T}_j - rank\widetilde{T}_{j-1}$$

and

$$\tilde{k}_0 = \min\{j : \tilde{r}_j = nq, j \geqslant 0\}.$$

Then the degree of the determinant of the pencil $z\widetilde{E} + \widetilde{A}$ is given by

$$\tilde{\delta} = rank\widetilde{T}_{\tilde{k}} - nq(\tilde{k}+1), \forall \tilde{k} \geqslant \tilde{k}_0.$$

We will show that there exists $k \geqslant k_0$ and $\tilde{k} \geqslant \tilde{k}_0$, such that $\delta = \tilde{\delta}$ ie we will prove that

$$rankT_k - n(k+1) = rank\widetilde{T}_{\tilde{k}} - nq(\tilde{k}+1). \tag{24}$$

The matrices T_k and $\widetilde{T}_{\tilde{k}}$ are of the same dimensions if and only if

$$n(q+k+1) = nq(\tilde{k}+2)$$

or equivalently

$$k - q\tilde{k} = q - 1. \tag{25}$$

The general solution of the above diophantine equation over the integers is

$$\begin{cases} k = cq + q - 1, \\ \tilde{k} = c, \end{cases} \quad c \in \mathbb{Z}. \tag{26}$$

Taking a closer look to the structure of the matrices, one can see that for k and $\tilde{k}$ as defined above and $c \in \mathbb{N}$

$$T_k = \widetilde{T}_{\tilde{k}}$$

and so

$$rankT_k = rank\widetilde{T}_{\tilde{k}}. \tag{27}$$

Choosing a large enough c such that $k \geqslant k_0$ and $\tilde{k} \geqslant \tilde{k}_0$ and taking into account (27), Eq. (24) becomes

$$-n(cq + q - 1 + 1) = -nq(c+1),$$

which obviously holds. So we have proven that the degrees of the determinants of $z\widetilde{E} + \widetilde{A}$ and $A(z)$ are equal.

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